

ESSENTIAL NORM ESTIMATES FOR THE $\bar{\partial}$ -NEUMANN OPERATOR ON CONVEX DOMAINS AND WORM DOMAINS

ŽELJKO ČUČKOVIĆ AND SÖNMEZ ŞAHUTOĞLU

ABSTRACT. In the paper we give a lower estimate for the essential norm of the $\bar{\partial}$ -Neumann operator on convex domains and worm domains of Diederich and Fornæss.

Let Ω be a bounded pseudoconvex domain in \mathbb{C}^n and $b\Omega$ denote the boundary of Ω . The space of square integrable $(0, q)$ -forms on Ω is denoted by $L^2_{(0,q)}(\Omega)$ for $0 \leq q \leq n$. In this paper we will only consider $(0, q)$ -forms instead of (p, q) -forms because the theory is independent of p . The operator $\bar{\partial} : L^2_{(0,q)}(\Omega) \rightarrow L^2_{(0,q+1)}(\Omega)$ is a closed, linear, and densely defined unbounded operator and it has a Hilbert space adjoint $\bar{\partial}^* : L^2_{(0,q+1)}(\Omega) \rightarrow L^2_{(0,q)}(\Omega)$. This is an important operator in complex analysis.

The $\bar{\partial}$ -Neumann operator, denoted by N_q , is the solution operator for the complex Laplacian $\square_q = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} : L^2_{(0,q)}(\Omega) \rightarrow L^2_{(0,q)}(\Omega)$. The $\bar{\partial}$ -Neumann operator is a self-adjoint bounded linear operator on $L^2_{(0,q)}(\Omega)$ and $\bar{\partial}^*N_q$ gives the solution operator for $\bar{\partial}$ with minimal norm. Sobolev regularity properties of N_q are important in several complex variables and have been widely studied. For a survey of such results we refer the reader to [BS99]. For more information about the $\bar{\partial}$ -Neumann operator we refer the reader to two excellent books on the subject [CS01, Str10].

Compactness of the $\bar{\partial}$ -Neumann operator is stronger than its global regularity [KN65]. There are potential theoretic (Property (P) of Catlin [Cat84] and Property (\tilde{P}) of McNeal [McN02]) as well as geometric ([MS07, Str08]) sufficient conditions for compactness. Yet, it is not clear if these conditions are also necessary in general. In case of convex domains compactness of N_q is well understood. Fu and Straube in [FS98] showed that for $1 \leq q \leq n$ the following conditions are equivalent: compactness of N_q , the domain satisfying Property (P_q), absence of q -dimensional varieties in the boundary of the domain, and compactness of the commutators $[P_{q-1}, \bar{z}_j]$ for $1 \leq j \leq n$ (here P_{q-1} is the Bergman projection on $(0, q-1)$ -forms). For more information about compactness of the $\bar{\partial}$ -Neumann problem and related topics we refer the reader to the survey [FS01] and the book [Str10].

The aim of this paper is to quantify the failure of compactness of the $\bar{\partial}$ -Neumann operator in terms of boundary geometry. As far as we know this is the first attempt in that direction.

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Let X and Y be two normed linear spaces and $T : X \rightarrow Y$ be a bounded linear operator. The *essential norm* of T , denoted by $\|T\|_e$, is defined as

$$\|T\|_e = \inf\{\|T - K\| : K : X \rightarrow Y \text{ is a compact operator}\}$$

where $\|\cdot\|$ denotes the operator norm.

The motivation for this paper came from a previous paper [ČŠ] in which we studied the essential norm estimates for a Hankel operator $H_\varphi = [\varphi, P]$ in terms of the behavior of the symbol φ on the discs in the boundary. Compactness of the $\bar{\partial}$ -Neumann operator is closely connected to compactness of Hankel operators (see [ČŠ12, ČŠ14]). We note that, it is still unclear if compactness of H_φ on $A^2(\Omega)$, the Bergman space on Ω , for all $\varphi \in C(\bar{\Omega})$ is sufficient for compactness of N . This is known as D'Angelo's question.

The plan of the paper is as follows: In the next section we will state the main result, Theorem 1, establishing a lower bound for the essential norm of N_q on convex domains in \mathbb{C}^n . Then we continue with a section devoted to Theorem 2, an application of our techniques to get a lower bound for the essential norm of the $\bar{\partial}$ -Neumann operator on the Diederich-Fornæss type worm domains. Finally, in the last section we present some basic facts about the essential norms of operators, the Proposition 1, and the proofs of Theorems 1 and 2.

THE MAIN RESULT

Throughout this paper $\|f\|$ will denote the L^2 norm of the function f . When we want to emphasize the domain we will denote the L^2 norm on Ω by $\|\cdot\|_\Omega$. Let us define $C_0^1(\bar{\Omega})$ to be the set of real-valued functions that are C^1 -smooth on $\bar{\Omega}$ and vanish on $b\Omega$. Let us also define

$$\alpha_\Omega = \sup \left\{ \frac{2 \int_\Omega \chi(z) dV(z)}{\|\nabla \chi\|} : \chi \in C_0^1(\bar{\Omega}) \text{ and } \chi \not\equiv 0 \right\}$$

where $\nabla \chi$ denotes the (real) gradient of χ . Let $r = (r_1, \dots, r_n)$. By $r > 0$ (respectively $r \geq 0$) we mean $r_j > 0$ (respectively $r_j \geq 0$) for $1 \leq j \leq n$. We will denote the polydisc in \mathbb{C}^n centered at w with polyradius $r > 0$ by $D(w, r) = \{z \in \mathbb{C}^n : |z_j - w_j| < r_j, 1 \leq j \leq n\}$. We use the convention $D(w, 0) = \{w\}$. We define $\beta_{D(w, 0)} = 0$ and

$$\beta_{D(w, r)} = \frac{\prod_{k=1}^n r_k}{\sqrt{\sum_{k=1}^n \frac{1}{r_k^2}}}$$

if $r > 0$.

We note that α_D is the square root of the *torsional rigidity* of D when D is a simply connected domain in \mathbb{C} . Physically, torsional rigidity of $D \subset \mathbb{C}$ is proportional to the discharge of a viscous fluid flowing through a pipe with the cross section D (see [PS51, pg 103]).

Theorem 1. *Let Ω be a bounded convex domain in \mathbb{C}^n and τ_Ω denote the diameter of Ω . Assume that q_Ω is the largest dimension of the (affine) analytic varieties in $b\Omega$.*

i. *If $q \geq q_\Omega = 0$ or $q > q_\Omega \geq 0$ then $\|N_q\|_e = 0$.*

ii. *If $1 \leq q \leq q_\Omega \leq n - 1$ then*

$$\|N_q\|_e \geq \frac{C(n, q_\Omega)}{\tau_\Omega^{2q_\Omega}} \sup \left\{ \beta_{D(w, r)}^2 : D(w, r) \text{ is } q_\Omega\text{-dimensional polydisc in } b\Omega \text{ with } r \geq 0 \right\}$$

where

$$C(n, q_\Omega) = \frac{(q_\Omega + 1)^{2q_\Omega + 2} (n - q_\Omega)^{2n - 2q_\Omega}}{(n + 1)^{2n + 2}} \frac{3^{q_\Omega - 1}}{2^{2q_\Omega + 1}}.$$

iii. *If $1 \leq q \leq q_\Omega = n - 1$ and Ω has C^1 -smooth boundary then*

$$\|N_q\|_e \geq \frac{(n - 1)!}{\pi^{n-1} \tau_\Omega^{2n-2}} \sup \left\{ \alpha_M^2 : M \text{ is an affine } (n - 1)\text{-dimensional variety in } b\Omega \right\}.$$

Let \mathbb{D} be the unit open disc in the complex plane and $\Delta(a, b) = \{a + b\xi : \xi \in \mathbb{D}\}$ where $a, b \in \mathbb{C}^n$. So $\Delta(a, b)$ is an (affine) disc in \mathbb{C}^n centered at a with radius $|b|$ and $\Delta(a, 0) = \{a\}$. Then iii. in Theorem 1 leads to the following corollary.

Corollary 1. *Let Ω be a bounded convex domain in \mathbb{C}^2 with C^1 -smooth boundary. Then*

$$\|N_1\|_e \geq \frac{r_{b\Omega}^4}{2\tau_\Omega^2}$$

where $r_{b\Omega} = \sup\{|b| : \Delta(a, b) \subset b\Omega\}$ and τ_Ω denote the diameter of Ω .

We note that the inequality in the corollary above is due to the following fact: for a convex domain $M \subset \mathbb{C}$ we have $\alpha_M \geq r_M \sqrt{\frac{V(M)}{2}}$ where r_M denotes the radius of the largest circle contained in M (see [PS51, pg 99-100]).

Remark 1. The essential norm of a self-adjoint operator is related to the essential spectrum (part of the spectrum that is the complement of the eigenvalues with finite multiplicities) of the operator. More precisely, if T is a self-adjoint operator and $\sigma_e(T)$ denotes the essential spectrum T , then $\|T\|_e = \sup\{|\lambda| : \lambda \in \sigma_e(T)\}$. Since N_q is self-adjoint our results give lower bound estimates for the radius of the essential spectrum of N_q in case the domain is bounded and convex.

APPLICATION TO WORM DOMAINS

Let us start by defining more general versions of Diederich-Fornæss worm domains. Let $r > 1, \beta > 0$, and

$$\rho_{\beta, r}(z_1, z_2) = \left| z_1 - e^{i2\beta \log |z_2|} \right| - 1 + \sigma(|z_2|^2 - r^2) + \sigma(1 - |z_2|^2)$$

where

$$\sigma(t) = \begin{cases} Me^{-1/t}, & t > 0 \\ 0, & t \leq 0 \end{cases}$$

for $M > 0$. Then for large enough M the domains

$$\Omega_{\beta,r} = \left\{ (z_1, z_2) \in \mathbb{C}^2 : \rho_{\beta,r}(z_1, z_2) < 0 \right\}$$

are smooth bounded and pseudoconvex (see [BŞ12, Proposition 1]). These domains have a total winding of $2\beta \log r$ and contain the annulus $A_r = \{\xi \in \mathbb{C} : 1 < |\xi| < r\}$.

Worm domains originally have been constructed to show that some smooth bounded pseudoconvex domains do not have Stein neighborhood basis for their closures [DF77]. However, they turned out to be a class of domains with irregular Bergman projections and $\bar{\partial}$ -Neumann operators [Bar92, Chr96, KP08]. Now they are considered one of the important classes of domains in several complex variables. We choose to work on these domains rather than the original worm domains because we can decouple the winding numbers from the size of the annuli.

In the next theorem we give a lower bound estimate for the essential norm of the $\bar{\partial}$ -Neumann operator on worm domains defined above.

Theorem 2. *Let $r > 1$ and $\beta > 0$. Then the $\bar{\partial}$ -Neumann operator on $\Omega_{\beta,r}$ has the following essential norm estimate*

$$\|N_1\|_e \geq \max \left\{ \left(\frac{\eta^2 + 1}{2} - \frac{\eta^2 - 1}{2 \log \eta} \right) \frac{\pi - 2\beta \log \eta}{\pi + 2\beta \log \eta} : 1 < \eta < \min\{e^{\pi/2\beta}, r\} \right\}.$$

It is interesting that the estimate in Theorem 2 depends on the winding number as well as the size of the annulus in the boundary. In contrast, the irregularity results of the $\bar{\partial}$ -Neumann operator on the worm domains depend on the winding number only [Bar92, BŞ12].

PROOFS

We will need the following lemmas for the proof of the theorems.

Lemma 1. *Let X and Y be two Hilbert spaces and $T : X \rightarrow Y$ be a bounded linear operator. Then*

$$\|T\|_e^2 = \|T^*\|_e^2 = \|T^*T\|_e = \|TT^*\|_e.$$

Proof. Let us define $\tilde{T} : X \oplus Y \rightarrow X \oplus Y$ by $\tilde{T}(x, y) = (0, Tx)$. Then $\|T\| = \|\tilde{T}\|$. First we will show that $\|T\|_e = \|\tilde{T}\|_e$. Let $K : X \rightarrow Y$ be a linear compact operator. Then $\|\tilde{T} - \tilde{K}\| = \|T - K\|$ where the linear compact operator $\tilde{K} : X \oplus Y \rightarrow X \oplus Y$ is defined by $\tilde{K}(x, y) = (0, Kx)$. Hence taking infimum over K implies that $\|\tilde{T}\|_e \leq \|T\|_e$.

To show the reverse inequality, let π_X and π_Y denote the projections from $X \oplus Y$ onto X and Y , respectively. Let $\tilde{K} : X \oplus Y \rightarrow X \oplus Y$ be a compact linear operator. Then the component

operators $\tilde{K}_1 = \pi_X \tilde{K}$ and $\tilde{K}_2 = \pi_Y \tilde{K}$ are compact. Let us define $\tilde{T}_2 = \pi_Y \tilde{T}$. That is, $\tilde{T}_2(x, y) = Tx$. Then

$$\begin{aligned} \|\tilde{T} - \tilde{K}\|^2 &= \|\tilde{K}_1\|^2 + \|\tilde{T}_2 - \tilde{K}_2\|^2 \\ &\geq \sup \left\{ \|Tx - \tilde{K}_2(x, y)\|^2 : \|x\|^2 + \|y\|^2 \leq 1 \right\} \\ &\geq \sup \left\{ \|Tx - \tilde{K}_2(x, 0)\|^2 : \|x\|^2 \leq 1 \right\} \\ &= \|T - K\|^2 \end{aligned}$$

where $K : X \rightarrow Y$ is a compact operator defined by $Kx = \tilde{K}_2(x, 0)$. Taking infimum over \tilde{K} we get $\|\tilde{T}\|_e \geq \|T\|_e$. Therefore, we showed that

$$(1) \quad \|T\|_e = \|\tilde{T}\|_e.$$

We will continue the proof with computing \tilde{T}^* . Let $x, u \in X$ and $y, v \in Y$. Then

$$\begin{aligned} \langle \tilde{T}^*(x, y), (u, v) \rangle &= \langle (x, y), \tilde{T}(u, v) \rangle \\ &= \langle (x, y), (0, Tu) \rangle \\ &= \langle (T^*y, 0), (u, v) \rangle. \end{aligned}$$

Hence $\tilde{T}^*(x, y) = (T^*y, 0)$, $\|T^*\| = \|\tilde{T}^*\|$ and, as was done earlier in the proof, one can show that $\|T^*\|_e = \|\tilde{T}^*\|_e$. Furthermore

$$\tilde{T}^* \tilde{T}(x, y) = \tilde{T}^*(0, Tx) = (T^*Tx, 0)$$

and

$$\tilde{T} \tilde{T}^*(x, y) = \tilde{T}(T^*y, 0) = (0, TT^*y).$$

Therefore,

$$(2) \quad \|T^*T\|_e = \|\tilde{T}^* \tilde{T}\|_e \text{ and } \|TT^*\|_e = \|\tilde{T} \tilde{T}^*\|_e.$$

Finally the fact that the Calkin algebra on a Hilbert space is a C^* -algebra (see, for example, [Con00, 5.6 Theorem]) implies that

$$\|\tilde{T}\|_e^2 = \|\tilde{T}^*\|_e^2 = \|\tilde{T}^* \tilde{T}\|_e = \|\tilde{T} \tilde{T}^*\|_e.$$

Now combining the equality above with equalities (1) and (2) we get

$$\|T\|_e^2 = \|T^*\|_e^2 = \|T^*T\|_e = \|TT^*\|_e.$$

Hence the proof of the lemma is complete. \square

Remark 2. We use the Lemma above to show that the lower estimate in [ČŠ, Theorem 2] is sharp, in case $\varphi(z_1, z_2) = \bar{z}_1$. First one can show that $H_{\bar{z}_1}^{\mathbb{D}^2}(z_1^j z_2^k) = H_{\bar{z}_1}^{\mathbb{D}}(z_1^j) z_2^k$ for all $j, k \geq 0$ and

$\langle H_{\bar{z}_1}^{\mathbb{D}} z_1^j, H_{\bar{z}_1}^{\mathbb{D}} z_1^k \rangle_{\mathbb{D}} = 0$ unless $j = k$ (the inner product is on \mathbb{D}). We use this fact in the equality below. In the computations below we denote the domain as a subscript unless it is \mathbb{D}^2 .

$$\begin{aligned} \left\| H_{\bar{z}_1}^{\mathbb{D}^2} \sum_{j,k=0}^{\infty} a_{jk} z_1^j z_2^k \right\|^2 &= \sum_{j,k=0}^{\infty} |a_{jk}|^2 \left\| H_{\bar{z}_1}^{\mathbb{D}} z_1^j \right\|_{\mathbb{D}}^2 \|z_2^k\|_{\mathbb{D}}^2 \\ &\leq \frac{1}{2} \sum_{j,k=0}^{\infty} |a_{jk}|^2 \|z_1^j\|_{\mathbb{D}}^2 \|z_2^k\|_{\mathbb{D}}^2 \\ &= \frac{1}{2} \left\| \sum_{j,k=0}^{\infty} a_{jk} z_1^j z_2^k \right\|^2. \end{aligned}$$

In the last inequality we used the fact that $\left\| H_{\bar{z}_1}^{\mathbb{D}} \right\| = 1/\sqrt{2}$ (see [OR16, Theorem 1]). Therefore, $\left\| H_{\bar{z}_1}^{\mathbb{D}^2} \right\|_{\mathbb{D}^2} \leq 1/\sqrt{2}$.

Next we will show that $\left\| H_{\bar{z}_1}^{\mathbb{D}^2} \right\|_e \geq 1/\sqrt{2}$. Since $\left(H_{\bar{z}_1}^{\mathbb{D}} \right)^* H_{\bar{z}_1}^{\mathbb{D}}$ is a self-adjoint compact operator and its norm equals $1/2$, there exists an eigenfunction $f \in A^2(\mathbb{D})$ such that

$$\left(H_{\bar{z}_1}^{\mathbb{D}} \right)^* H_{\bar{z}_1}^{\mathbb{D}} f(z_1) = \frac{f(z_1)}{2}.$$

Then for $k \geq 0$ we have

$$\left(H_{\bar{z}_1}^{\mathbb{D}^2} \right)^* H_{\bar{z}_1}^{\mathbb{D}^2} (f(z_1) z_2^k) = \left(\left(H_{\bar{z}_1}^{\mathbb{D}} \right)^* H_{\bar{z}_1}^{\mathbb{D}} (f(z_1)) \right) z_2^k = \frac{f(z_1) z_2^k}{2}.$$

That is, $1/2$ is an eigenvalue for $\left(H_{\bar{z}_1}^{\mathbb{D}^2} \right)^* H_{\bar{z}_1}^{\mathbb{D}^2}$ with infinite multiplicity. Then $1/2$ is in the essential spectrum of $\left(H_{\bar{z}_1}^{\mathbb{D}^2} \right)^* H_{\bar{z}_1}^{\mathbb{D}^2}$ (see [Con90, Chapter XI, 4.6 Proposition]) and

$$\left\| \left(H_{\bar{z}_1}^{\mathbb{D}^2} \right)^* H_{\bar{z}_1}^{\mathbb{D}^2} \right\|_e \geq \frac{1}{2}.$$

So

$$\frac{1}{\sqrt{2}} \leq \sqrt{\left\| \left(H_{\bar{z}_1}^{\mathbb{D}^2} \right)^* H_{\bar{z}_1}^{\mathbb{D}^2} \right\|_e} = \left\| H_{\bar{z}_1}^{\mathbb{D}^2} \right\|_e \leq \left\| H_{\bar{z}_1}^{\mathbb{D}^2} \right\| \leq \frac{1}{\sqrt{2}}.$$

Therefore, $\left\| H_{\bar{z}_1}^{\mathbb{D}^2} \right\| = \left\| H_{\bar{z}_1}^{\mathbb{D}^2} \right\|_e = 1/\sqrt{2}$.

Remark 3. Let Ω be a bounded pseudoconvex domain in \mathbb{C}^n . Let us define the operator $M_{\bar{\partial}\varphi} : A^2(\Omega) \rightarrow L^2_{(0,1)}(\Omega)$ as $M_{\bar{\partial}\varphi} f = f \bar{\partial}\varphi$. We note that $\|M_{\bar{\partial}\bar{z}_k}\|_e = 1$ for $1 \leq k \leq n$. This can be seen as follows: Let $\{f_j\}$ be an orthonormal basis of $A^2(\Omega)$. Then using the fact that compact operators turn weakly convergent sequences into convergent sequence we conclude that

$$\lim_{j \rightarrow \infty} \|M_{\bar{\partial}\bar{z}_k} f_j - K f_j\| = \lim_{j \rightarrow \infty} \|M_{\bar{\partial}\bar{z}_k} f_j\| = \lim_{j \rightarrow \infty} \|f_j\| = 1$$

for any compact operator $K : A^2(\Omega) \rightarrow L^2_{(0,1)}(\Omega)$. Hence, $1 \leq \|M_{\bar{\partial}\bar{z}_k}\|_e \leq \|M_{\bar{\partial}\bar{z}_k}\| = 1$.

Let $\bar{\partial}^* N_{1,a}$ denote the restriction of $\bar{\partial}^* N_1$ onto $A^2_{(0,1)}(\Omega)$, the $(0,1)$ -forms with square integrable holomorphic coefficients. Then one can show that

$$\|H_{\bar{z}_k}\|_e \leq \|\bar{\partial}^* N_{1,a}\|_e \|M_{\bar{\partial}\bar{z}_k}\|_e$$

for $k = 1, 2, \dots, n$. The fact that $\|M_{\bar{\partial}\bar{z}_k}\|_e = 1$ implies that

$$\|\bar{\partial}^* N_1\|_e \geq \|\bar{\partial}^* N_{1,a}\|_e \geq \max \{ \|H_{\bar{z}_k}\|_e : k = 1, 2, \dots, n \}.$$

Therefore, in case $\Omega = \mathbb{D}^2$ we get (see Corollary 3)

$$\|N_1\|_e = \|\bar{\partial}^* N_1\|_e^2 \geq \|\bar{\partial}^* N_{1,a}\|_e^2 \geq \|H_{\bar{z}_1}\|_e^2 = \frac{1}{2}.$$

Comparing this estimate to Siqi Fu's result in [Fu07, pg 729] about the bottom of the spectrum of \square_1 shows that our estimate is not sharp on the bidisc. Indeed, the bottom of the spectrum of \square_1 on the bidisc is $j_{0,1}^2/4 \approx 1.44576576$ where $j_{0,1} \approx 2.4048$ is the first positive zero of the Bessel function of order zero. So $\|N_1\|_e = 4/j_{0,1}^2 \approx 0.69 > 1/2$.

Lemma 2. *Let X, Y , and Z be Hilbert spaces, and $T_1 : X \rightarrow Y$ and $T_2 : X \rightarrow Z$ be bounded linear operators. Assume that $T : X \rightarrow Y \oplus Z$ be given by $T = T_1 \oplus T_2$. Then*

$$\|T\|_e^2 = \|T_1\|_e^2 + \|T_2\|_e^2.$$

Proof. Let π_Y and π_Z denote the projections from $Y \oplus Z$ onto Y and Z , respectively. Assume that $K : X \rightarrow Y \oplus Z$ is a compact operator. Then $K_1 = \pi_Y K$ and $K_2 = \pi_Z K$ are compact and

$$\|T - K\|^2 = \|T_1 - K_1\|^2 + \|T_2 - K_2\|^2 \geq \|T_1\|_e^2 + \|T_2\|_e^2.$$

Then taking infimum over K we get

$$\|T\|_e^2 \geq \|T_1\|_e^2 + \|T_2\|_e^2.$$

Similarly, if $K_1 : X \rightarrow Y$ and $K_2 : X \rightarrow Z$ are compact operators then $K = K_1 \oplus K_2 : X \rightarrow Y \oplus Z$ is compact and

$$\|T\|_e^2 \leq \|T - K\|^2 = \|T_1 - K_1\|^2 + \|T_2 - K_2\|^2.$$

Taking infimum over K_1 and K_2 we get

$$\|T\|_e^2 \leq \|T_1\|_e^2 + \|T_2\|_e^2.$$

Therefore, $\|T\|_e^2 = \|T_1\|_e^2 + \|T_2\|_e^2$. □

Now we will prove a more precise version of [ÇŞ14, Lemma 1]. We note that $K^2_{(0,q)}(\Omega)$ denotes the $\bar{\partial}$ -closed $(0,q)$ -forms on Ω .

Lemma 3. *Let Ω be a bounded pseudoconvex domain in \mathbb{C}^n for $n \geq 2$ and $g \in K_{(0,q+1)}^2(\Omega)$ where $1 \leq q \leq n-1$. Then there exist $g_j \in K_{(0,q)}^2(\Omega)$ for $1 \leq j \leq n$ such that*

$$g = \sum_{j=1}^n g_j \wedge d\bar{z}_j \text{ and } \sum_{j=1}^n \|g_j\|^2 \leq \|g\|^2.$$

Proof. Let $1 \leq q \leq n-1$ and

$$f = \sum'_{|J|=q} f_J d\bar{z}_J = \bar{\partial}^* N_{q+1} g \in L_{(0,q)}^2(\Omega).$$

The symbol $\sum'_{|J|=q}$ above denotes the summation over strictly increasing index J . That is, $J = j_1 j_2 \cdots j_q$ with $j_1 < j_2 < \cdots < j_q$. Let \vee denote the adjoint of the exterior multiplication. That is, if f is a $(0, q)$ -form $d\bar{z}_j \vee f$ is a $(0, q-1)$ -form such that $\langle h \wedge d\bar{z}_j, f \rangle = \langle h, d\bar{z}_j \vee f \rangle$ for all $h \in L_{(0,q-1)}^2(\Omega)$. We define $f_j = d\bar{z}_j \vee f$ for $1 \leq j \leq n$. Then one can show that

$$f_j = \sum'_{|I|=q-1} f_{jI} d\bar{z}^I \text{ for } 1 \leq j \leq n \text{ and } f = \frac{1}{q} \sum_{j=1}^n d\bar{z}_j \wedge f_j.$$

Every f_J appears in q different f_j 's for $J = jI$. The decomposition above was observed by Jeffery McNeal and it has appeared in [Str10, pg. 75]. Then

$$\bar{\partial} f_j = \sum'_{|I|=q-1} \bar{\partial} f_{jI} \wedge d\bar{z}^I = \sum'_{|I|=q} F_j^I d\bar{z}^I$$

where each F_j^I is a sum of at most q terms of the form $\frac{\partial f_{jI}}{\partial \bar{z}_k}$ because each term appears at most once. Now we use the fact that $(x_1 + \cdots + x_q)^2 \leq q(x_1^2 + \cdots + x_q^2)$ for real numbers x_1, \dots, x_q to conclude that

$$\|\bar{\partial} f_j\|^2 \leq q \sum'_{|I|=q-1} \sum_{k=1}^n \left\| \frac{\partial f_{jI}}{\partial \bar{z}_k} \right\|^2$$

for all k 's. We note that q^2 appears on the second equality below because each $\frac{\partial f_{jI}}{\partial \bar{z}_k}$ appears q many times as $\frac{\partial f_I}{\partial \bar{z}_k}$. Then we use [Str10, Corollary 2.13] to get

$$\begin{aligned} \sum_{j=1}^n \|\bar{\partial} f_j\|^2 &\leq q \sum_{j=1}^n \sum'_{|I|=q-1} \sum_{k=1}^n \left\| \frac{\partial f_{jI}}{\partial \bar{z}_k} \right\|^2 \\ &= q^2 \sum'_{|J|=q} \sum_{k=1}^n \left\| \frac{\partial f_J}{\partial \bar{z}_k} \right\|^2 \\ &\leq q^2 (\|\bar{\partial} f\|^2 + \|\bar{\partial}^* f\|^2) \\ &= q^2 \|g\|^2. \end{aligned}$$

Let us define $g_j = \frac{(-1)^{q-1}}{q} \bar{\partial} f_j$. Then $\sum_{j=1}^n \|g_j\|^2 \leq \|g\|^2$ and

$$g = \bar{\partial} \bar{\partial}^* N_{q+1} g = \bar{\partial} f = \frac{(-1)^{q-1}}{q} \sum_{j=1}^n \bar{\partial} f_j \wedge d\bar{z}_j = \sum_{j=1}^n g_j \wedge d\bar{z}_j.$$

Hence the proof of the lemma is complete. \square

Lemma 4. *Let Ω be a bounded pseudoconvex domain in \mathbb{C}^n and $1 \leq q \leq n-1$. Then*

$$\|\bar{\partial}^* N_{q+1}\|_e \leq \|\bar{\partial}^* N_q\|_e.$$

Proof. Let $K_{(0,q)}^2(\Omega)$ denote the $\bar{\partial}$ -closed square integrable $(0, q)$ -forms. Assume that $f \in K_{(0,q+1)}^2(\Omega)$. Then by Lemma 3 for $1 \leq k \leq n$ there exists $f_k \in K_{(0,q)}^2(\Omega)$ such that

$$f = \sum_{k=1}^n f_k \wedge d\bar{z}_k \text{ and } \sum_{k=1}^n \|f_k\|^2 \leq \|f\|^2.$$

Assume that $\alpha_q = \|\bar{\partial}^* N_q\|_e$. Then for $\varepsilon > 0$ there exists a compact operator K_q^ε on $L_{(0,q)}^2(\Omega)$ such that

$$\|\bar{\partial}^* N_q - K_q^\varepsilon\| < \alpha_q + \varepsilon.$$

Let us define

$$S_{q+1}f = \sum_{k=1}^n \bar{\partial}^* N_q(f_k) \wedge d\bar{z}_k \text{ and } K_{q+1}^\varepsilon f = \sum_{k=1}^n K_q^\varepsilon(f_k) \wedge d\bar{z}_k.$$

Then

$$\|S_{q+1}f - K_{q+1}^\varepsilon f\|^2 \leq \sum_{k=1}^n \|\bar{\partial}^* N_q f_k - K_q^\varepsilon f_k\|^2 \leq \|\bar{\partial}^* N_q - K_q^\varepsilon\|^2 \sum_{k=1}^n \|f_k\|^2.$$

That is, $\|S_{q+1} - K_{q+1}^\varepsilon\| \leq \|\bar{\partial}^* N_q - K_q^\varepsilon\|$ as $\sum_{k=1}^n \|f_k\|^2 \leq \|f\|^2$. We note that on $K_{(0,q+1)}^2(\Omega)$ we have

$$\bar{\partial}^* N_{q+1} = (I - P_{q+1})S_{q+1}$$

because $\bar{\partial}^* N_{q+1}$ is the canonical solution operator for $\bar{\partial}$ and S_{q+1} is a solution operator. Then

$$\bar{\partial}^* N_{q+1} - \tilde{K}_{q+1}^\varepsilon = (I - P_{q+1})(S_{q+1} - K_{q+1}^\varepsilon)$$

where $\tilde{K}_{q+1}^\varepsilon = (I - P_{q+1})K_{q+1}^\varepsilon$ is a compact operator. Hence

$$\|\bar{\partial}^* N_{q+1} - \tilde{K}_{q+1}^\varepsilon\| \leq \|S_{q+1} - K_{q+1}^\varepsilon\| \leq \|\bar{\partial}^* N_q - K_q^\varepsilon\| \leq \alpha_q + \varepsilon.$$

We complete the proof of the lemma by letting $\varepsilon \rightarrow 0$. \square

Corollary 2. *Let Ω be a bounded pseudoconvex domain in \mathbb{C}^n and $1 \leq q \leq n-1$. Then*

$$\|N_{q+1}\|_e \leq \|N_q\|_e.$$

Furthermore, $\frac{1}{\sqrt{2}}\|N_1\|_e \leq \|N_0\|_e \leq \|N_1\|_e$.

Proof. Let $1 \leq q \leq n - 1$. Lemma 4 implies that

$$\|\bar{\partial}^* N_{q+2}\|_e \leq \|\bar{\partial}^* N_{q+1}\|_e \leq \|\bar{\partial}^* N_q\|_e.$$

Now we will use Range's formula [Ran84, FS01],

$$N_q = (\bar{\partial}^* N_q)^* (\bar{\partial}^* N_q) + (\bar{\partial}^* N_{q+1}) (\bar{\partial}^* N_{q+1})^*$$

and the fact that the ranges of the operator $(\bar{\partial}^* N_q)^* (\bar{\partial}^* N_q)$ and $(\bar{\partial}^* N_{q+1}) (\bar{\partial}^* N_{q+1})^*$ are in $\text{Ker}(\bar{\partial})$ and $\text{Im}(\bar{\partial}^*)$, respectively. Then since $L^2_{(0,q)}(\Omega) = \text{Ker}(\bar{\partial}) \oplus \text{Im}(\bar{\partial}^*)$ we can apply Lemmas 1 and 2 to conclude that

$$(3) \quad \|N_q\|_e^2 = \|\bar{\partial}^* N_q\|_e^4 + \|\bar{\partial}^* N_{q+1}\|_e^4 \text{ and } \|N_{q+1}\|_e^2 = \|\bar{\partial}^* N_{q+1}\|_e^4 + \|\bar{\partial}^* N_{q+2}\|_e^4.$$

Then Lemma 4 implies that

$$\|N_{q+1}\|_e^2 = \|\bar{\partial}^* N_{q+1}\|_e^4 + \|\bar{\partial}^* N_{q+2}\|_e^4 \leq \|\bar{\partial}^* N_{q+1}\|_e^4 + \|\bar{\partial}^* N_q\|_e^4 \leq \|N_q\|_e^2.$$

Therefore, $\|N_{q+1}\|_e \leq \|N_q\|_e$.

In case $q = 0$, Range's formula is $N_0 = \bar{\partial}^* N_1 (\bar{\partial}^* N_1)^*$. Then Lemma 1 implies that

$$\|N_0\|_e = \|\bar{\partial}^* N_1\|_e^2.$$

Furthermore, using the fact that $\|N_1\|_e^2 = \|\bar{\partial}^* N_1\|_e^4 + \|\bar{\partial}^* N_2\|_e^4 \leq 2\|\bar{\partial}^* N_1\|_e^4$ we conclude that,

$$\frac{1}{\sqrt{2}} \|N_1\|_e \leq \|N_0\|_e \leq \|N_1\|_e.$$

□

Range's formula implies that

$$N_1 = (\bar{\partial}^* N_1)^* (\bar{\partial}^* N_1) + (\bar{\partial}^* N_2) (\bar{\partial}^* N_2)^*.$$

If the domain is bounded and in \mathbb{C}^2 then $\bar{\partial}^* N_2$ is compact (the proof of this is contained in the proof of Theorem 2.1 in [AS15]). Then Lemma 1 implies that $\|N_1\|_e = \|\bar{\partial}^* N_1\|_e^2$. On the other hand, $\|N_0\|_e = \|\bar{\partial}^* N_1\|_e^2$. Therefore, we have the following corollary.

Corollary 3. *Let Ω be a bounded pseudoconvex domain in \mathbb{C}^2 . Then*

$$\|N_0\|_e = \|N_1\|_e = \|\bar{\partial}^* N_1\|_e^2.$$

Lemma 5. *Let Ω be a convex domain in \mathbb{C}^n for $n \geq 2$, $p \in b\Omega$. Assume that M_1 and M_2 are two analytic varieties in the $b\Omega$ and $p \in M_1 \cap M_2$. Then the convex hull of $M_1 \cup M_2$ is an affine analytic variety in $b\Omega$.*

Proof. We will use the following fact: Any analytic variety in the boundary of a convex domain in \mathbb{C}^n is contained in affine analytic variety in the boundary of the domain. This fact was proven for analytic discs in [ČŞ09, Lemma 2] (see also [FS98] as well as [BS92, McN92]). The same

proof works for higher dimensional varieties in the boundary of a convex domain in \mathbb{C}^n as well. Without loss of generality assume that

- i. $\Omega \subset \{(z_1, \dots, z_n) \in \mathbb{C}^n : \operatorname{Re}(z_n) < 0\}$ and $0 \in b\Omega$,
- ii. M_1 and M_2 are affine analytic varieties in $b\Omega$ such that $0 \in M_1 \cap M_2$.

Since M_1 and M_2 are affine analytic varieties (using the fact that tangent space of a complex manifold in $b\Omega$ is in the complex tangent space of $b\Omega$) we conclude that

$$M_1 \cup M_2 \subset \{(z_1, \dots, z_n) \in \mathbb{C}^n : z_n = 0\}.$$

Let M denote the convex hull of $M_1 \cup M_2$. Then M is contained in the complex hyperplane $\{(z_1, \dots, z_n) \in \mathbb{C}^n : z_n = 0\}$. If M is not an $(n-1)$ -dimensional complex manifold, by applying rotation in the first $(n-1)$ -variables if necessary, we may assume that $M \subset \{(z_1, \dots, z_n) \in \mathbb{C}^n : \operatorname{Re}(z_{n-1}) = z_n = 0\}$. Invoking the fact that M_1 and M_2 are complex manifolds again we conclude that $M \subset \{(z_1, \dots, z_n) \in \mathbb{C}^n : z_{n-1} = z_n = 0\}$. Using this argument, finitely many times, we reach the conclusion that M is an affine analytic variety in $b\Omega$. \square

Lemma 6. *Let Ω be a smooth bounded pseudoconvex or a bounded convex domain in \mathbb{C}^n , $p \in b\Omega$, and $k_z(w) = K(w, z) / \sqrt{K(z, z)}$ where K is the Bergman kernel of Ω . Then $k_z \rightarrow 0$ weakly as $z \rightarrow p$.*

Proof. Without loss of generality we may assume that $0 \in \Omega$. Then $A^\infty(\overline{\Omega})$, the space of functions holomorphic on Ω and smooth up to the boundary, is dense in $A^2(\Omega)$. In case of bounded convex domain this can be seen as follows: if $f \in A^2(\Omega)$ then the function $f_\delta(z) = f((1-\delta)z)$ is holomorphic on a neighborhood of $\overline{\Omega}$ for any $\delta > 0$ and $f_\delta \rightarrow f$ in L^2 norm as $\delta \rightarrow 0^+$. In case Ω is smooth bounded and pseudoconvex this is a result of Catlin [Cat80, Theorem 3.2.1].

Let $\varepsilon > 0$ be given. Then there exists $f_\delta \in A^\infty(\overline{\Omega})$ such that $\|f - f_\delta\| < \varepsilon$. Then

$$|\langle f, k_z \rangle| \leq |\langle f - f_\delta, k_z \rangle| + |\langle f_\delta, k_z \rangle| \leq \|f - f_\delta\| + |\langle f_\delta, k_z \rangle| < \varepsilon + |\langle f_\delta, k_z \rangle|$$

However, we note that $\langle f_\delta, k_z \rangle = f_\delta(z) / \sqrt{K(z, z)} \rightarrow 0$ as $z \rightarrow p$ because $K(z, z) \rightarrow \infty$ as $z \rightarrow p$ (see [JP93, Theorem 6.1.17] and [Pfl75]) and f_δ is bounded. Since ε was arbitrary we conclude that $\lim_{z \rightarrow p} \langle f, k_z \rangle = 0$ for any $f \in A^2(\Omega)$. That is, $k_z \rightarrow 0$ weakly as $z \rightarrow p$. \square

Proposition 1. *Let Ω be a bounded convex domain in \mathbb{C}^n and τ_Ω denote the diameter of Ω . Assume that $b\Omega$ contains a non-trivial analytic variety and q is the largest dimension of the analytic varieties in $b\Omega$.*

- i. *If $1 \leq q \leq n-1$ then*

$$\|\bar{\partial}^* N_q\|_e \geq \frac{c(n, q)}{\tau_\Omega^q} \sup \left\{ \beta_{D(w, r)} : D(w, r) \text{ is } q\text{-dimensional polydisc in } b\Omega \text{ with } r \geq 0 \right\}$$

where

$$c(n, q) = \frac{(q+1)^{q+1}(n-q)^{n-q}}{(n+1)^{n+1}} \left(\frac{3^{q-1}}{2^{2q+1}} \right)^{1/2}.$$

ii. If $q = n - 1$ and Ω has C^1 -smooth boundary then

$$\|\bar{\partial}^* N_{n-1}\|_e \geq \sqrt{\frac{(n-1)!}{\pi^{n-1}}} \frac{1}{\tau_\Omega^{n-1}} \sup \{\alpha_M : M \text{ is an affine } (n-1)\text{-dimensional variety in } b\Omega\}.$$

Remark 4. Let $r = (r_1, \dots, r_n)$ with $r_j > 0$ for $1 \leq j \leq n$ and $D(0, r) = \{z \in \mathbb{C}^n : |z_j| < r_j, 1 \leq j \leq n\}$ be a polydisc. Using (5) in the proof of Proposition 1 one can estimate

$$\alpha_{D(0,r)} \geq \sqrt{\frac{3^{n-1}\pi^n}{2^{2n-1}}} \frac{\prod_{k=1}^n r_k}{\sqrt{\sum_{k=1}^n \frac{1}{r_k^2}}}.$$

Proof of Proposition 1. Let us prove i. first. Assume that $1 \leq q \leq n - 1$ is the largest dimension of analytic varieties in the boundary of Ω . Then Lemma 5 implies that there is an affine q -dimensional analytic variety in $b\Omega$. By applying a holomorphic affine transformation if necessary, we may assume that

- a. $b\Omega$ contains a nontrivial q -dimensional polydisc $M = D(0, r)$ where $1 \leq q \leq n - 1$ and $r = (r_1, \dots, r_q)$ for $r_j > 0$ for all j ,
- b. $M \times \{0\} \subset b\Omega \cap \{z \in \mathbb{C}^n : z_{q+1} = \dots = z_n = 0\}$,
- c. there are no analytic discs in $b\Omega$ transversal to M .

Let $M_\lambda = \lambda M \subset M$ for $0 < \lambda < 1$ and let us denote $z' = (z_1, \dots, z_q)$ and $z'' = (z_{q+1}, \dots, z_n)$ for $z = (z_1, \dots, z_n)$. Assume that

$$\Omega \subset \{z' \in \mathbb{C}^q : \|z'\| < \tau_\Omega\} \times \{z'' \in \mathbb{C}^{n-q} : \|z''\| < \tau_\Omega, \operatorname{Re}(z_n) > 0\}.$$

Let $\{p_j\} \subset \Omega_s = \{z'' \in \mathbb{C}^{n-q} : (0, z'') \in \Omega\}$ be a sequence (to be determined later) converging to the origin and

$$\tilde{f}_j(z'') = \frac{K_{\Omega_s}(z'', p_j)}{\sqrt{K_{\Omega_s}(p_j, p_j)}}.$$

Then $\|\tilde{f}_j\|_{\Omega_s} = 1$ and Lemma 6 implies that the sequence $\{f_j\}$ converges to zero weakly as $p_j \rightarrow 0 \in b\Omega_s$. One can show that convexity of Ω implies that $M_\lambda \times (1 - \lambda)\Omega_s \subset \Omega$.

We use [Bł13, Theorem 1] (a version of Ohsawa-Takegoshi Theorem [OT87]) repeatedly q times with $D = \{z \in \mathbb{C} : |z| < \tau_\Omega\}$ and the fact that $c_D(0) = \sqrt{\pi K_D(0)} = 1/\tau_\Omega$ to extend \tilde{f}_j 's to Ω , we call the extension f_j , such that

$$(4) \quad \|f_j\|_\Omega^2 \leq \pi^q \tau_\Omega^{2q}$$

and $f_j(0, z'') = \tilde{f}_j(z'')$. Let

$$\chi_j(\xi) = \frac{2}{\pi \lambda^2 r_j^2} \left(1 - \frac{|\xi|^2}{\lambda^2 r_j^2} \right) \text{ for } \xi \in \mathbb{C}.$$

Then $\chi_j \in C^\infty(\mathbb{C})$ and $\chi_j(\xi) = 0$ for $|\xi| = \lambda r_j$. Using polar coordinates, one can compute that

$$\begin{aligned} \int_{\{|\xi| < \lambda r_j\}} \chi_j(\xi) dV(\xi) &= 1, \\ \int_{\{|\xi| < \lambda r_j\}} |\chi_j(\xi)|^2 dV(\xi) &= \frac{4}{3\pi\lambda^2 r_j^2}, \\ \int_{\{|\xi| < \lambda r_j\}} |(\chi_j)_\xi(\xi)|^2 dV(\xi) &= \frac{2}{\pi\lambda^4 r_j^4}. \end{aligned}$$

Let

$$F_j = f_j d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_q \text{ and } \Phi = \chi d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_q$$

where $\chi(z') = \chi_1(z_1) \cdots \chi_q(z_q)$. Assume that ϑ denotes the formal adjoint on $\bar{\partial}$. Then

$$\vartheta\Phi = - \sum_{k=1}^q (-1)^{k-1} \frac{\partial\chi}{\partial z_k} d\bar{z}_1 \wedge \cdots \wedge \widehat{d\bar{z}_k} \wedge \cdots \wedge d\bar{z}_q$$

where $\widehat{d\bar{z}_k}$ means that $d\bar{z}_k$ is missing. Then

$$\begin{aligned} \int_{M_\lambda} |\chi_{z_j}(z')|^2 dV(z') &= \int_{\{|\xi| < \lambda r_j\}} |(\chi_j)_{z_j}(\xi)|^2 dV(\xi) \prod_{k \neq j} \int_{\{|\xi| < \lambda r_k\}} |\chi_k(\xi)|^2 dV(\xi) \\ &= \frac{2}{\pi\lambda^4 r_j^4} \prod_{k \neq j} \frac{4}{3\pi\lambda^2 r_k^2} \\ &= \frac{2^{2q-1}}{3^{q-1}\pi^q \lambda^{2q+2}} \frac{1}{r_j^2} \prod_{k=1}^q \frac{1}{r_k^2}. \end{aligned}$$

Then

$$(5) \quad \|\vartheta\Phi\|_{M_\lambda}^2 = \frac{2^{2q-1}}{3^{q-1}\pi^q \lambda^{2q+2}} \frac{\sum_{k=1}^q \frac{1}{r_k^2}}{\prod_{k=1}^q r_k^2}.$$

Now we will derive an integration by parts formula for F_j 's. Let $G = g d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_q$ where g is a square integrable holomorphic function on Ω . Then

$$\bar{\partial}^* N_q G = \sum_{k=1}^q g_k d\bar{z}_1 \wedge \cdots \wedge \widehat{d\bar{z}_k} \wedge \cdots \wedge d\bar{z}_q + H$$

where g_k 's are square integrable functions and H is a $(0, q-1)$ -form that includes terms containing $d\bar{z}_j$ for some $j \geq q+1$ (in case of $q=1$ the form H is zero). Since G is a $\bar{\partial}$ -closed form $G = \bar{\partial}\bar{\partial}^* N_q G$. Then by comparing the types of forms in G and $\bar{\partial}\bar{\partial}^* N_q G$ we conclude that

$$\bar{\partial}\bar{\partial}^* N_q G = \sum_{k=1}^q (-1)^{k-1} \frac{\partial g_k}{\partial \bar{z}_k} d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_q.$$

Then

$$\begin{aligned}
\int_{M_\lambda} \langle \vartheta \Phi, \bar{\partial}^* N_q G \rangle &= - \sum_{k=1}^q (-1)^{k-1} \int_{M_\lambda} \frac{\partial \chi}{\partial z_k} \bar{g}_k \\
&= \sum_{k=1}^q (-1)^{k-1} \int_{M_\lambda} \chi \frac{\bar{\partial} g_k}{\partial \bar{z}_k} \\
&= \int_{M_\lambda} \langle \Phi, \bar{\partial} \bar{\partial}^* N_q G \rangle.
\end{aligned}$$

Now we apply the equality above to F_j 's. For a fixed j we have

$$\int_{M_\lambda} \langle \Phi, \bar{\partial} \bar{\partial}^* N_q F_j \rangle = \int_{M_\lambda} \langle \vartheta \Phi, \bar{\partial}^* N_q F_j \rangle.$$

Then (we emphasize the variables in the first line below) using the fact that $\int_{M_\lambda} \chi(z') dV(z') = 1$ in the first equality below we get

$$\begin{aligned}
\overline{f_j(0, z'')} &= \overline{f_j(0, z'')} \int_{M_\lambda} \chi(z') dV(z') \\
&= \int_{M_\lambda} \chi(z') \overline{f_j(z', z'')} dV(z') \\
&= \int_{M_\lambda} \langle \Phi, \bar{\partial} \bar{\partial}^* N_q F_j \rangle \\
&= \int_{M_\lambda} \langle \vartheta \Phi, \bar{\partial}^* N_q F_j \rangle \\
&\leq \| \vartheta \Phi \|_{M_\lambda} \| \bar{\partial}^* N_q F_j \|_{M_\lambda}.
\end{aligned}$$

We take the norm square of both sides and integrate in z'' variables on $(1 - \lambda)\Omega_s$ to get

$$(6) \quad \| \bar{\partial}^* N_q F_j \| \geq \| \bar{\partial}^* N_q F_j \|_{M_\lambda \times (1-\lambda)\Omega_s} \geq \frac{\| f_j \|_{(1-\lambda)\Omega_s}}{\| \vartheta \Phi \|_{M_\lambda}}.$$

Now we will compute $\| f_j \|_{(1-\lambda)\Omega_s}$. Let us apply the reproducing property of $K_{(1-\lambda)\Omega_s}(p_j, \cdot)$ to $K_{\Omega_s}(\cdot, p_j)$ on $(1 - \lambda)\Omega_s$.

$$\begin{aligned}
K_{\Omega_s}(p_j, p_j) &= \int_{(1-\lambda)\Omega_s} K_{(1-\lambda)\Omega_s}(p_j, z'') K_{\Omega_s}(z'', p_j) dV(z'') \\
&\leq \| K_{(1-\lambda)\Omega_s}(p_j, \cdot) \|_{(1-\lambda)\Omega_s} \| K_{\Omega_s}(\cdot, p_j) \|_{(1-\lambda)\Omega_s} \\
&= \sqrt{K_{(1-\lambda)\Omega_s}(p_j, p_j)} \| K_{\Omega_s}(\cdot, p_j) \|_{(1-\lambda)\Omega_s}.
\end{aligned}$$

We used the Cauchy-Schwarz inequality for the inequality on the second line. Namely, we get

$$\frac{K_{\Omega_s}(p_j, p_j)}{K_{(1-\lambda)\Omega_s}(p_j, p_j)} \leq \frac{\| K_{\Omega_s}(\cdot, p_j) \|_{(1-\lambda)\Omega_s}^2}{K_{\Omega_s}(p_j, p_j)}.$$

Hence if we write the right hand side above in terms of f_j and use

$$K_{(1-\lambda)\Omega_s}(p_j, p_j) = \frac{1}{(1-\lambda)^{2(n-q)}} K_{\Omega_s} \left(\frac{p_j}{1-\lambda}, \frac{p_j}{1-\lambda} \right)$$

we get

$$(7) \quad \|f_j\|_{(1-\lambda)\Omega_s}^2 \geq \frac{K_{\Omega_s}(p_j, p_j)}{K_{(1-\lambda)\Omega_s}(p_j, p_j)} = (1-\lambda)^{2(n-q)} \frac{K_{\Omega_s}(p_j, p_j)}{K_{\Omega_s} \left(\frac{p_j}{1-\lambda}, \frac{p_j}{1-\lambda} \right)}.$$

Let $\delta > 0, \alpha = \frac{\lambda\delta}{1-\lambda}$, and $\rho_\delta = \delta\rho_0$ where $\rho_0 = (0, \dots, 0, 1)$. We note that $\rho_\delta \in \Omega_s$ for small $\delta > 0$. Let us define $T_\alpha(z) = z + (0, \dots, 0, \alpha)$. Since Ω_s is convex, we can choose U small enough so that $T_\alpha(\Omega_s \cap U) \subset \Omega_s$. Since there is no analytic disc in the boundary of Ω_s through ρ_0 , [FS98, Proposition 3.2] implies that ρ_0 is a peak point and in turn [Nik02, Theorem 2] implies that for $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that $0 < \delta < \delta_\varepsilon$ implies that

$$K_{\Omega_s}(\rho_\delta, \rho_\delta) \geq (1-\varepsilon)K_{\Omega_s \cap U}(\rho_\delta, \rho_\delta).$$

Furthermore, we have

$$K_{\Omega_s \cap U}(\rho_\delta, \rho_\delta) = K_{T_\alpha(\Omega_s \cap U)}(\rho_{\delta+\alpha}, \rho_{\delta+\alpha}) \geq K_{\Omega_s}(\rho_{\delta+\alpha}, \rho_{\delta+\alpha}).$$

Therefore,

$$(8) \quad K_{\Omega_s}(\rho_\delta, \rho_\delta) \geq (1-\varepsilon)K_{\Omega_s}(\rho_{\delta+\alpha}, \rho_{\delta+\alpha}).$$

Now we choose $\{p_j\}$ as $p_j = \rho_{1/j} = (0, \dots, 0, 1/j)$. Note that $\delta + \alpha = \delta/(1-\lambda)$. Let us choose $\delta = 1/j$. Then $\rho_{\delta+\alpha} = \rho_\delta/(1-\lambda) = p_j/(1-\lambda)$. Furthermore, the fact that ε is arbitrary and (8) imply that

$$\liminf_{j \rightarrow \infty} \frac{K_{\Omega_s}(p_j, p_j)}{K_{\Omega_s} \left(\frac{p_j}{1-\lambda}, \frac{p_j}{1-\lambda} \right)} \geq 1.$$

Then (5) and (7) imply that

$$(9) \quad \liminf_{j \rightarrow \infty} \frac{\|f_j\|_{(1-\lambda)\Omega_s}^2}{\|\vartheta\Phi\|_{M_\lambda}^2} \geq \lambda^{2q+2}(1-\lambda)^{2n-2q} \frac{3^{q-1}\pi^q \prod_{k=1}^q r_k^2}{2^{2q-1} \sum_{k=1}^q \frac{1}{r_k^2}}.$$

Now we want to find

$$\sup \left\{ \lambda^{2q+2}(1-\lambda)^{2n-2q} : 0 \leq \lambda \leq 1 \right\}.$$

One can compute that the maximum of $f(\lambda) = \lambda^{2q+2}(1-\lambda)^{2n-2q}$ over the closed interval $[0, 1]$ is attained at $\lambda = (q+1)/(n+1)$ and it is

$$\frac{(q+1)^{2q+2}(n-q)^{2n-2q}}{(n+1)^{2n+2}}.$$

For the rest of the proof of i. we fix $\lambda = (q+1)/(n+1)$. For any $\varepsilon > 0$ given we choose a compact operator $K_\varepsilon : K_{(0,q)}^2(\Omega) \rightarrow L_{(0,q)}^2(\Omega)$ such that

$$\|\bar{\partial}^* N_q - K_\varepsilon\| < \|\bar{\partial}^* N_q\|_e + \varepsilon.$$

Let us choose a subsequence of $\{F_j\}$, if necessary, so that $F_j \rightarrow F$ weakly. One can show that $F = f d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_q$ for some $f \in A^2(\Omega)$. One can also show that $\|F\| \leq \liminf_{j \rightarrow \infty} \|F_j\|$. Furthermore, $K_\varepsilon(F_j - F) \rightarrow 0$ and

$$\|\bar{\partial}^* N_q\|_e \geq \limsup_{j \rightarrow \infty} \frac{\|\bar{\partial}^* N_q(F_j - F)\|}{\|F_j - F\|} - \varepsilon \geq \limsup_{j \rightarrow \infty} \frac{\|\bar{\partial}^* N_q(F_j - F)\|}{2\|F_j\|} - \varepsilon.$$

We note that $F|_{(1-\lambda)\Omega_s} = 0$. This can be seen as follows: Let K_Ω denote the Bergman kernel of Ω and $z \in (1-\lambda)\Omega_s$. We remind the reader that Lemma 6 implies that $\{f_j\}$ converges to zero weakly. Then

$$F(z) = \langle F, K_\Omega(\cdot, z) \rangle = \lim_{j \rightarrow \infty} \langle F_j, K_\Omega(\cdot, z) \rangle = \lim_{j \rightarrow \infty} f_j(z) = 0.$$

Hence $\|f_j - f\|_{(1-\lambda)\Omega_s} = \|f_j\|_{(1-\lambda)\Omega_s}$ and using (4),(6),(9) we get

$$\|\bar{\partial}^* N_q\|_e^2 \geq \limsup_{j \rightarrow \infty} \frac{\|f_j\|_{(1-\lambda)\Omega_s}^2}{2^2 \pi^q \tau_\Omega^{2q} \|\vartheta \Phi\|_{M_\lambda}^2} - \varepsilon.$$

Since ε is arbitrary we get

$$\|\bar{\partial}^* N_q\|_e^2 \geq \frac{(q+1)^{2q+2} (n-q)^{2n-2q}}{(n+1)^{2n+2}} \frac{3^{q-1} \prod_{k=1}^q r_k^2}{2^{2q+1} \tau_\Omega^{2q} \sum_{k=1}^q \frac{1}{r_k^2}} = \frac{(c(n,q))^2}{\tau_\Omega^{2q}} \beta_{D(w,r)}^2.$$

This finishes the proof of the first part.

Now we will prove the case $q = n-1$. In this case we denote $z = (z', z_n) \in \mathbb{C}^n$ where $z' = (z_1, \dots, z_{n-1}) \in \mathbb{C}^{n-1}$. By using translation and rotation if necessary, without loss of generality, we may assume that

- i. $\Omega \subset \{z' \in \mathbb{C}^{n-1} : \|z'\| < \tau_\Omega\} \times \{z_n \in \mathbb{C} : |z_n| < \tau_\Omega, \operatorname{Re}(z_n) > 0\},$
- ii. $M = \{z' \in \mathbb{C}^{n-1} : (z', 0) \in b\Omega\}$ is $(n-1)$ -dimensional affine variety.

Since in this case we assume that Ω has C^1 -smooth boundary for $\varepsilon > 0$ there exists a wedge

$$W_{\pi-\varepsilon}^{r_0} = \left\{ r e^{i\theta} \in \mathbb{C} : 0 \leq r < r_0, |\theta| < \frac{\pi-\varepsilon}{2} \right\}$$

such that $M \times W_{\pi-\varepsilon}^{r_0} \subset \Omega$. We choose

$$f_j(z) = \frac{1}{2^j z_n^{\alpha_i}}$$

where $\alpha_j = 1 - 2^{-2j-1}$. Then $f_j \rightarrow 0$ weakly in $L^2(\Omega)$ and using i. above one can compute that

$$\|f_j\|^2 \leq \frac{1}{2^{2j}} \int_{\|z'\| < \tau_\Omega} dV(z') \int_{-\pi/2}^{\pi/2} d\theta \int_0^{\tau_\Omega} \frac{dr}{r^{2\alpha_j-1}} = \pi \omega_{2n-2} \tau_\Omega^{2n-2} \tau_\Omega^{2-2\alpha_j}$$

where ω_{2n-2} denotes the volume of the unit ball in \mathbb{R}^{2n-2} . We also need to compute $\|f_j\|_{W_{\pi-\varepsilon}^{r_0}}$

$$\|f_j\|_{W_{\pi-\varepsilon}^{r_0}}^2 \geq \frac{1}{2^{2j}} \int_{-(\pi-\varepsilon)/2}^{(\pi-\varepsilon)/2} d\theta \int_0^{r_0} \frac{dr}{r^{2\alpha_j-1}} = (\pi - \varepsilon) r_0^{2-2\alpha_j}.$$

Let

$$F_j = f_j d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_{n-1} \text{ and } \Phi = \chi d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_{n-1}$$

where $\chi \in C^1(\bar{M})$ not identically zero and $\chi = 0$ on the boundary of M . Then for $z_n \in W_{\pi-\varepsilon}^{r_0}$ we have

$$\begin{aligned} \frac{1}{2^j z_n^{\alpha_j}} \int_M \chi(z') dV(z') &= \int_M \chi(z') f_j(z) dV(z') \\ &= \int_M \langle \Phi, \bar{\partial}^* N_{n-1} F_j \rangle \\ &= \int_M \langle \vartheta \Phi, \bar{\partial}^* N_{n-1} F_j \rangle \\ &\leq \|\vartheta \Phi\|_M \|\bar{\partial}^* N_{n-1} F_j\|_M. \end{aligned}$$

We note that, unlike the previous case, in the computations above χ is not necessarily radially symmetric. Then by integrating in the last variable we get

$$(10) \quad \|\bar{\partial}^* N_{n-1} F_j\| \geq \|\bar{\partial}^* N_{n-1} F_j\|_{M \times W_{\pi-\varepsilon}^{r_0}} \geq \frac{\int_M \chi(z') dV(z')}{\|\vartheta \Phi\|_M} \|f_j\|_{W_{\pi-\varepsilon}^{r_0}}.$$

Furthermore, the fact that $\|\vartheta \Phi\|_M = \|\nabla \chi\|_M / 2$ and the estimate $\|f_j\|_{W_{\pi-\varepsilon}^{r_0}}^2 \geq (\pi - \varepsilon) r_0^{2-2\alpha_j}$ imply that

$$\|\bar{\partial}^* N_{n-1} F_j\| \geq \|\bar{\partial}^* N_{n-1} F_j\|_{M \times W_{\pi-\varepsilon}^{r_0}} \geq \frac{2 \int_M \chi(z') dV(z')}{\|\nabla \chi\|_M} \sqrt{(\pi - \varepsilon) r_0^{2-2\alpha_j}}.$$

Finally,

$$\begin{aligned} \|\bar{\partial}^* N_{n-1}\|_e &\geq \limsup_{j \rightarrow \infty} \frac{\|\bar{\partial}^* N_{n-1} F_j\|}{\|F_j\|} \\ &\geq \frac{2 \int_M \chi(z') dV(z')}{\|\nabla \chi\|_M} \limsup_{j \rightarrow \infty} \sqrt{\frac{(\pi - \varepsilon) r_0^{2-2\alpha_j}}{\pi \omega_{2n-2} \tau_\Omega^{2n-2} \tau_\Omega^{2-2\alpha_j}}} \\ &= \frac{2 \int_M \chi(z') dV(z')}{\|\nabla \chi\|_M} \sqrt{\frac{\pi - \varepsilon}{\pi \omega_{2n-2} \tau_\Omega^{2n-2}}}. \end{aligned}$$

Since ε was arbitrary we get

$$\|\bar{\partial}^* N_{n-1}\|_e \geq \frac{2 \int_M \chi(z') dV(z')}{\|\nabla \chi\|_M} \frac{1}{\sqrt{\omega_{2n-2}} \tau_\Omega^{n-1}}.$$

Therefore, taking supremum over χ and using the fact that $\omega_{2n-2} = \pi^{n-1}/(n-1)!$ we get

$$\|\bar{\partial}^* N_{n-1}\|_e \geq \alpha_M \sqrt{\frac{(n-1)!}{\pi^{n-1}}} \frac{1}{\tau_\Omega^{n-1}}$$

for every M . □

Remark 5. The proof of iii. of Theorem 1 can be modified to work on product domains even though they do not have C^1 -smooth boundary. Let U be a bounded pseudoconvex domain in \mathbb{C}^n and $\Omega_r = \{z \in \mathbb{C} : |z| < r\} \times U$. Then the proof of iii. in Theorem 1 modified to work on Ω_r with $M = U$ implies the following essential norm estimate of N_n on Ω_r :

$$\|N_n\|_e \geq \alpha_U^2 \frac{n!}{\pi^n \tau_{\Omega_r}^{2n}}.$$

Combining this estimate with Hörmander's estimate for the norm of N_n on Ω_r we get

$$\alpha_U^2 \frac{n!}{\pi^n \tau_{\Omega_r}^{2n}} \leq \|N_n\| \leq e \frac{\tau_{\Omega_r}^2}{n}$$

Then letting r go to zero (hence $\tau_{\Omega_r} \rightarrow \tau_U$) we get

$$\alpha_U \leq \frac{\tau_U^{n+1}}{n} \sqrt{\frac{e\pi^n}{(n-1)!}}.$$

When $U = \mathbb{D}$ this inequality is not sharp because $4\sqrt{e\pi} > \alpha_{\mathbb{D}} = \sqrt{\pi/2}$ (see Remark 6). In case U is a simply connected domain in the complex plane it is known that

$$\alpha_U \leq \frac{V(U)}{\sqrt{2\pi}}.$$

This is known as Saint-Venants inequality (see [PS51, pg 121] and also [Mak66, BFL14, FK15]).

Now we are ready to prove Theorem 1.

Proof of Theorem 1. The proof of i. follows from Corollary 2 and the fact that if the boundary of a bounded convex domain Ω does not contain any analytic variety of dimension greater than or equal to $q \geq 1$ then N_q on Ω is compact [FS98, Theorem 1.1].

To prove ii. let us assume that $1 \leq q \leq q_\Omega \leq n-1$. The fact that $\bar{\partial}N_q$ is compact for $q \geq q_\Omega + 1$ together with i. in Proposition 1 and the first equation in (3) imply that

$$\begin{aligned} \|N_{q_\Omega}\|_e &= \|\bar{\partial}^* N_{q_\Omega}\|_e^2 \\ &\geq \frac{C(n, q_\Omega)}{\tau_\Omega^{2q_\Omega}} \sup \left\{ \beta_{D(w,r)}^2 : D(w,r) \text{ is } q_\Omega\text{-dimensional polydisc in } b\Omega \text{ with } r \geq 0 \right\} \end{aligned}$$

where

$$C(n, q_\Omega) = \frac{(q_\Omega + 1)^{2q_\Omega+2} (n - q_\Omega)^{2n-2q_\Omega}}{(n+1)^{2n+2}} \frac{3^{q_\Omega-1}}{2^{2q_\Omega+1}}.$$

Then Corollary 2 implies that

$$\|N_q\|_e \geq \frac{C(n, q_\Omega)}{\tau_\Omega^{2q_\Omega}} \sup \left\{ \beta_{D(w,r)}^2 : D(w,r) \text{ is } q_\Omega\text{-dimensional polydisc in } b\Omega \text{ with } r \geq 0 \right\}$$

for $1 \leq q \leq q_\Omega$.

The proof of iii. is similar to the proof of ii. The only difference is that we use the essential norm estimate for $\bar{\partial}^* N_{n-1}$ in ii. in Proposition 1. \square

The following lemma will be used to compute α_Ω in case Ω is an annulus in \mathbb{C} .

Lemma 7. *Let Ω be a C^1 -smooth bounded domain in \mathbb{C} and $u \in C^2(\Omega) \cap C(\bar{\Omega})$ be the real valued function satisfying the following properties: $u = 0$ on $b\Omega$ and $u_{z\bar{z}} = -1$ on Ω . Then*

$$\alpha_\Omega = \frac{\int_\Omega u(z) dV(z)}{\|u_z\|} = \|u_z\|.$$

Proof. Using the fact that u is real valued together with integration by parts we get

$$\|u_z\|^2 = \int_\Omega u_z(z) u_{\bar{z}}(z) dV(z) = - \int_\Omega u(z) u_{z\bar{z}}(z) dV(z) = \int_\Omega u(z) dV(z).$$

Also one can check that $\|\nabla u\| = 2\|u_z\|$. Then $\alpha_\Omega \geq \frac{\int_\Omega u(z) dV(z)}{\|u_z\|} = \|u_z\|$.

To get the converse. Let $f \in C^2(\Omega) \cap C(\bar{\Omega})$ be a real valued function that vanishes on $b\Omega$ and $f \not\equiv 0$. Then

$$\int_\Omega f(z) dV(z) = - \int_\Omega f(z) u_{z\bar{z}}(z) dV(z) = \int_\Omega f_z(z) u_{\bar{z}}(z) dV(z) \leq \|f_z\| \|u_z\|.$$

That is, $\frac{\int_\Omega f(z) dV(z)}{\|f_z\|} \leq \|u_z\|$. Taking supremum over f we get $\alpha_\Omega \leq \|u_z\|$. \square

Remark 6. Let \mathbb{D}_r be the open disc with radius r . Then one can compute $\alpha_{\mathbb{D}_r} = \sqrt{\pi/2} r^2$ because in this case $u(z) = r^2 - |z|^2$.

Now let us compute α_{A_r} for the annulus $A_r = \{z \in \mathbb{C} : 1 < |z| < r\}$. The function

$$u(z) = r^2 - |z|^2 - \frac{r^2 - 1}{\log r} (\log r - \log |z|)$$

satisfies the conditions in the lemma. That is, $u_{z\bar{z}} = -1$ on A_r and $u = 0$ on bA_r . Then

$$u_z = -\bar{z} + \frac{r^2 - 1}{\log r} \frac{1}{2z}$$

and one can compute that

$$\|u_z\|^2 = \frac{\pi}{2} \left(r^4 - 1 - \frac{(r^2 - 1)^2}{\log r} \right).$$

We note that this is P' in [PS51, pg 103]. Therefore Lemma 7 implies that

$$(11) \quad \alpha_{A_r} = \sqrt{\frac{\pi}{2} \left(r^4 - 1 - \frac{(r^2 - 1)^2}{\log r} \right)}.$$

Now we are ready to prove Theorem 2.

Proof of Theorem 2. Let us denote $A_a^b = \{\zeta \in \mathbb{C} : a < |\zeta| < b\}$ and assume that $1 < \eta < \min\{e^{\pi/2\beta}, r\}$. Since $2\beta \log \eta \in (0, \pi)$ for every $0 < \varepsilon < \pi - 2\beta \log \eta$ there exists $\delta > 0$ such that we can put a wedge with angle $\pi - 2\beta \log \eta - \varepsilon$ and radius $\delta > 0$ in the domain that is perpendicular to $A_{1+\varepsilon}^{\eta-\varepsilon}$. More precisely,

$$W_{\pi-2\beta \log \eta - \varepsilon}^\delta \times A_{1+\varepsilon}^{\eta-\varepsilon} \subset \Omega_{\beta,r} \cap \{(z_1, z_2) \in \mathbb{C}^2 : 1 < |z_2| < \eta\}$$

We can also put $\Omega_{\beta,r} \cap \{(z_1, z_2) \in \mathbb{C}^2 : 1 < |z_2| < \eta\}$ in a similar product space. Hence we have

$$W_{\pi-2\beta \log \eta - \varepsilon}^\delta \times A_{1+\varepsilon}^{\eta-\varepsilon} \subset \Omega_{\beta,r} \cap \{(z_1, z_2) \in \mathbb{C}^2 : 1 < |z_2| < \eta\} \subset W_{\pi+2\beta \log \eta}^2 \times A_1^\eta$$

We use the same sequence of functions

$$f_j(z_1, z_2) = \frac{1}{2^j z_1^{\alpha_j}}$$

as in the proof of Theorem 1. Let χ_η be a function independent of z_1 such that $\chi_\eta(z_2) = 1$ if $1 \leq |z_2| \leq \eta$ and $\chi_\eta(z_2) = 0$ otherwise. Then

$$\|f_j\|_{W_{\pi-2\beta \log \eta - \varepsilon}^\delta}^2 = (\pi - 2\beta \log \eta - \varepsilon) \delta^{2-2\alpha_j} \text{ and } \|\chi_\eta f_j\|^2 \leq \pi(\pi + 2\beta \log \eta)(\eta^2 - 1) 2^{2-2\alpha_j}.$$

Let $\chi \in C^1(\overline{A_{1+\varepsilon}^{\eta-\varepsilon}})$ be a real valued function such that $\chi \equiv 0$ on the boundary of $A_{1+\varepsilon}^{\eta-\varepsilon}$ and $\int_{A_{1+\varepsilon}^{\eta-\varepsilon}} \chi(z_2) dV(z_2) = 1$. We think of χ as a function of z_2 . Then by similar computations as in (10) for $F_j = \chi_\eta f_j d\bar{z}_2$ we get

$$\|\bar{\partial}^* N_1(\chi_\eta f_j d\bar{z}_2)\| \geq \frac{2\|f_j\|_{W_{\pi-2\beta \log \eta - \varepsilon}^\delta}}{\|\nabla \chi\|_{A_{1+\varepsilon}^{\eta-\varepsilon}}} = \frac{2}{\|\nabla \chi\|_{A_{1+\varepsilon}^{\eta-\varepsilon}}} \delta^{1-\alpha_j} \sqrt{\pi - 2\beta \log \eta - \varepsilon}.$$

Then

$$\begin{aligned}
 \|\bar{\partial}^* N_1\|_e &\geq \limsup_{j \rightarrow \infty} \frac{\|\bar{\partial}^* N_1(\chi_\eta f_j d\bar{z}_2)\|}{\|\chi_\eta f_j\|} \\
 &\geq \limsup_{j \rightarrow \infty} \frac{2}{\|\nabla \chi\|_{A_{1+\varepsilon}^{\eta-\varepsilon}}} \frac{\sqrt{\pi - 2\beta \log \eta - \varepsilon}}{\sqrt{\pi(\pi + 2\beta \log \eta)(\eta^2 - 1)}} \left(\frac{\delta}{2}\right)^{1-\alpha_j} \\
 &= \frac{2}{\|\nabla \chi\|_{A_{1+\varepsilon}^{\eta-\varepsilon}}} \frac{\sqrt{\pi - 2\beta \log \eta - \varepsilon}}{\sqrt{\pi(\pi + 2\beta \log \eta)(\eta^2 - 1)}}.
 \end{aligned}$$

Therefore, if we let $\varepsilon \rightarrow 0$ and take supremum over χ we get

$$\|\bar{\partial}^* N_1\|_e \geq \alpha_{A_\eta} \frac{\sqrt{\pi - 2\beta \log \eta}}{\sqrt{\pi(\pi + 2\beta \log \eta)(\eta^2 - 1)}}.$$

Using the fact that $\|N_1\|_e = \|\bar{\partial}^* N_1\|_e^2$ on domains in \mathbb{C}^2 we get

$$\|N_1\|_e \geq \frac{\alpha_{A_\eta}^2}{\pi(\eta^2 - 1)} \frac{\pi - 2\beta \log \eta}{\pi + 2\beta \log \eta}.$$

Then (11) implies that

$$\|N_1\|_e \geq \left(\frac{\eta^2 + 1}{2} - \frac{\eta^2 - 1}{2 \log \eta} \right) \frac{\pi - 2\beta \log \eta}{\pi + 2\beta \log \eta}.$$

We complete the proof by taking maximum of the left hand side for $1 < \eta < \min\{e^{\pi/\beta}, r\}$. \square

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E-mail address: Zeljko.Cuckovic@utoledo.edu, Sonmez.Sahutoglu@utoledo.edu

UNIVERSITY OF TOLEDO, DEPARTMENT OF MATHEMATICS & STATISTICS, TOLEDO, OH 43606, USA